OXFORD UNIVERSITY

MATHEMATICS, JOINT SCHOOLS AND COMPUTER SCIENCE

WEDNESDAY 4 NOVEMBER 2009

Time allowed: $2\frac{1}{2}$ hours

For candidates applying for Mathematics, Mathematics & Statistics, Computer Science, Mathematics & Computer Science, or Mathematics & Philosophy

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Write your name, test centre (where you are sitting the test), Oxford college (to which you have applied or been assigned) and your proposed course (from the list above) in BLOCK CAPITALS.

NOTE: Separate sets of instructions for both candidates and test supervisors are provided, which should be read carefully before beginning the test.

NAME:

TEST CENTRE:

OXFORD COLLEGE (if known):

DEGREE COURSE:

DATE OF BIRTH:

FOR TEST SUPERVISORS USE ONLY:

[ ] **Tick here if special arrangements were made for the test.**

Please either include details of special provisions made for the test and the reasons for these in the space below or securely attach to the test script a letter with the details.

Signature of Invigilator ______________________

FOR OFFICE USE ONLY:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Total

1. For **ALL APPLICANTS**.

For each part of the question on pages 3–7 you will be given four possible answers, just one of which is correct. Indicate for each part **A**–**J** which answer (a), (b), (c), or (d) you think is correct with a tick (✓) in the corresponding column in the table below. *Please show any rough working in the space provided between the parts.*

	(a)	(b)	(c)	(d)
A				
B				
C				
D				
E				
F				
G				
H				
I				
J				

A. The smallest value of

$$I(a) = \int_0^1 \left(x^2 - a\right)^2 \, dx,$$

as a varies, is

(a) $\frac{3}{20}$, (b) $\frac{4}{45}$, (c) $\frac{7}{13}$, (d) 1.

$$I(a) = \int_0^1 (x^2 - a)^2 \, dx$$

$$= \int_0^1 x^4 - 2ax^2 + a^2 \, dx$$

$$= \left[\frac{x^5}{5} - \frac{2ax^3}{3} + a^2 x \right]_0^1$$

$$= \frac{1}{5} - \frac{2a}{3} + a^2 - 0$$

$$= \left(a - \frac{1}{3}\right)^2 - \frac{1}{9} + \frac{1}{5}$$

$$= \left(a - \frac{1}{3}\right)^2 + \frac{4}{45}$$

smallest value of $I(a)$ is when $\left(a - \frac{1}{3}\right)^2 = 0$ and $I(a) = \frac{4}{45}$

B. The point on the circle

$$x^2 + y^2 + 6x + 8y = 75,$$

which is closest to the origin, is at what distance from the origin?

(a) 3, (b) 4, (c) 5, (d) 10.

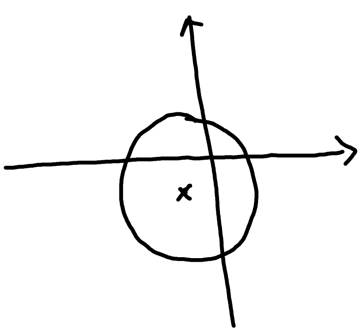
$(x+3)^2 + (y+4)^2 - 9 - 16 = 75$

$(x+3)^2 + (y+4)^2 = 100$

centre : $(-3, -4)$

radius : $\sqrt{100} = 10$

closest point to origin : radius from centre $(-3, -4)$, through origin which meets the circle at $(3, 4)$



distance $= \sqrt{3^2 + 4^2}$

$= 5$

Turn Over

C. Given a real constant c, the equation

$$x^4 = (x-c)^2$$

has four real solutions (including possible repeated roots) for

(a) $c \leqslant \frac{1}{4}$, (b) $-\frac{1}{4} \leqslant c \leqslant \frac{1}{4}$, (c) $c \leqslant -\frac{1}{4}$, (d) all values of c.

$x^2 = x - c$ or $x^2 = c - x$

$x^2 - x + c = 0$ $\quad$ $x^2 + x - c = 0$

$b^2 - 4ac$: $\quad 1 - 4c \quad\quad 1 + 4c$

For $x^4 = (x-c)^2$ to have 4 real solutions, each $x^2 = x - c$ and $x^2 = c - x$ needs to have 2 real solutions, so their discriminants must be $\geqslant 0$

$1 - 4c > 0 \quad\quad 1 + 4c \geqslant 0$

$\frac{1}{4} \geqslant c \quad\quad c \geqslant -\frac{1}{4}$

$-\frac{1}{4} \leq c \leq \frac{1}{4}$

D. The smallest positive integer n such that

$$1 - 2 + 3 - 4 + 5 - 6 + \cdots + (-1)^{n+1} n \geqslant 100,$$

is

(a) 99, (b) 101, (c) 199, (d) 300.

$1 - 2 + 3 - 4 + 5 - 6 + \ldots + (-1)^{n+1} n \geqslant 100$

$(-1) + (-1) + (-1) + \ldots$

$(-1) \times \left(\frac{n-1}{2}\right) + n \geqslant 100$

$\frac{1-n}{2} + n \geqslant 100$

$1 - n + 2n \geqslant 200$

$n \geqslant 199$

smallest positive integer is 199

E. In the range $0 \leqslant x < 2\pi$, the equation

$$2^{\sin^2 x} + 2^{\cos^2 x} = 2$$

(a) has 0 solutions;
(b) has 1 solution;
(c) has 2 solutions;
(d) holds for all values of x.

$-1 \leq \sin x \leq 1$ $\quad\quad$ $-1 \leq \cos x \leq 1$

$0 \leq \sin^2 x \leq 1$ $\quad\quad$ $0 \leq \cos^2 x \leq 1$

$2^0 \leq 2^{\sin^2 x} \leq 2^1$ $\quad\quad$ $2^0 \leq 2^{\cos^2 x} \leq 2^1$

$1 \leq 2^{\sin^2 x} \leq 2$ $\quad\quad$ $1 \leq 2^{\cos^2 x} \leq 2$

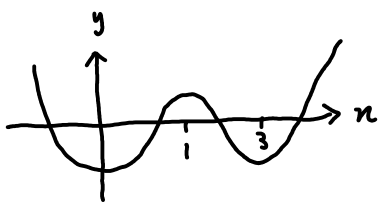
for $2^{\sin^2 x} + 2^{\cos^2 x} = 2$, both $\sin^2 x$ and $\cos^2 x$ must equal 0.
however $\sin^2 x + \cos^2 x = 1$ and if $\sin^2 x = 1$, $\cos^2 x = 0$
$\therefore$ the equation has 0 solutions

F. The equation in x

$$3x^4 - 16x^3 + 18x^2 + k = 0$$

has four real solutions
(a) when $-27 < k < 5$;
(b) when $5 < k < 27$;
(c) when $-27 < k < -5$;
(d) when $-5 < k < 0$

let $3x^4 - 16x^3 + 18x^2 + k = y$



$\frac{dy}{dx} = 12x^3 - 48x^2 + 36x = 0$

$12x(x^2 - 4x + 3) = 0$

$12x(x-3)(x-1) = 0$

turning points at $x = 0$ (min), $x = 1$ (max), $x = 3$ (min) (coefficient of $x^4 > 0$)

$x = 0$, $y = k$ $\quad$ $x = 1$, $y = 5 + k$ $\quad$ $x = 3$, $y = 27 + k$

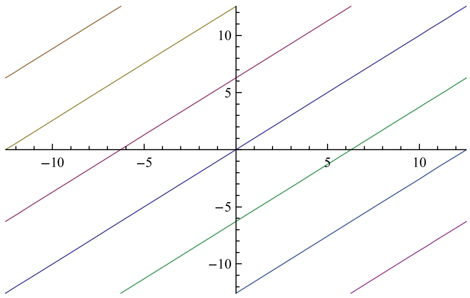
For four real solutions, all of the following conditions must be valid: The values of y at $x = 0$ and $x = 3$ must be less than 0, while at $x = 1$, the y value must be greater than 0

$k < 0$, $5 + k > 0$, $-27 + k < 0$

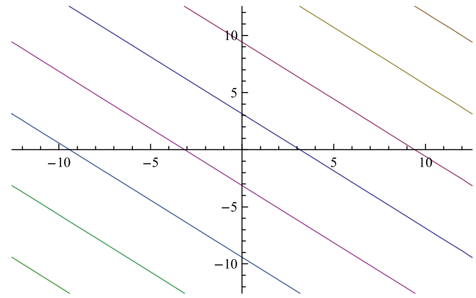
$\therefore -5 < k < 0$

Turn Over

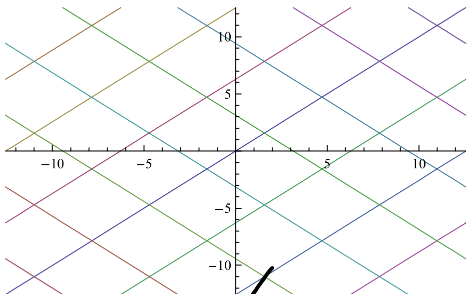
G. The graph of all those points (x, y) in the xy-plane which satisfy the equation $\sin y = \sin x$ is drawn in



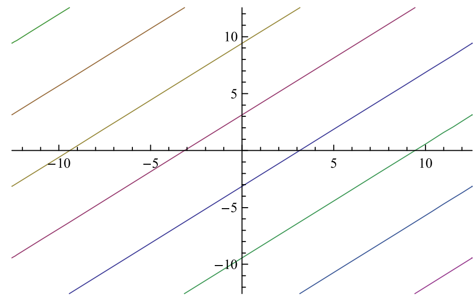
(a)



(b)



(c) ✓



(d)

when $x=0$ and $y=0$, $\sin y = \sin x = 0$, eliminating b and d
when $x=\pi$ and $y=0$, $\sin y = \sin x = 0$, eliminating a

H. When the trapezium rule is used to estimate the integral

$$\int_0^1 2^x \, dx$$

by dividing the interval $0 \leqslant x \leqslant 1$ into N subintervals the answer achieved is

(a) $\frac{1}{2N}\left\{1 + \frac{1}{2^{1/N}+1}\right\}$, ✓(b) $\frac{1}{2N}\left\{1 + \frac{2}{2^{1/N}-1}\right\}$,

(c) $\frac{1}{N}\left\{1 - \frac{1}{(2^{1/N}-1)}\right\}$, (d) $\frac{1}{2N}\left\{\frac{5}{2^{1/N}+1} - 1\right\}$.

$$\int_0^1 2^x dx = \frac{1}{2N}\left[2^0 + 2^1 + 2\underbrace{\left(2^{\frac{1}{N}} + 2^{\frac{2}{N}} + \ldots + 2^{\frac{N-1}{N}}\right)}_{\text{geometric series}}\right]$$

$$= \frac{1}{2N}\left[1 + 2 + 2\left[\frac{2^{\frac{1}{N}}\left(\left(2^{\frac{1}{N}}\right)^{N-1} - 1\right)}{2^{\frac{1}{N}} - 1}\right]\right]$$

$$= \frac{1}{2N}\left[3 + 2\left(\frac{2^1 - 2^{\frac{1}{N}}}{2^{\frac{1}{N}} - 1}\right)\right]$$

$$= \frac{1}{2N}\left[3 - 2\left(\frac{2^{\frac{1}{N}} - 1 - 1}{2^{\frac{1}{N}} - 1}\right)\right]$$

$$= \frac{1}{2N}\left[3 - 2 + \frac{2}{2^{\frac{1}{N}} - 1}\right]$$

$$= \frac{1}{2N}\left(1 + \frac{2}{{}^{\frac{1}{N}} - 1}\right)$$

I. The polynomial

$$n^2x^{2n+3} - 25nx^{n+1} + 150x^7$$

has $x^2 - 1$ as a factor
(a) for no values of n;
(b) for $n = 10$ only;
(c) for $n = 15$ only;
(d) for $n = 10$ and $n = 15$ only.

when $x = 1$
$n^2 - 25n + 150 = 0$
$(n-15)(n-10) = 0$
$n = 15\ ,\ n = 10$

when $x = -1$
$n^2(-1)^{2n+3} - 25n(-1)^{n+1} + (-1)(150) = 0$
$2n+3$ is always odd
$-n^2 - 25n(-1)^{n+1} - 150 = 0$

if n is odd:
$-n^2 - 25n - 150 = 0$
$(n+15)(n+10) = 0$
$n = -15$

if n is even:
$-n^2 + 25n - 150 = 0$
$(n-15)(n-10) = 0$
$n = 10$

$\therefore$ the polynomial has $x^2 - 1$ as a factor for $n = 10$ only.

J. The number of *pairs* of *positive integers* x, y which solve the equation

$$x^3 + 6x^2y + 12xy^2 + 8y^3 = 2^{30}$$

is

(a) 0, (b) 2^6, (c) $2^9 - 1$, (d) $2^{10} + 2$.

$x^3 + 6x^2y + 12xy^2 + 8y^3 = 2^{30}$
$(x+2y)^3 = (2^{10})^3$
$x + 2y = 2^{10}$

x must be even for y to be an integer. x cannot be 0 or 2^{10}.
$\therefore$ There are $\frac{2^{10}}{2} - 1 = 2^9 - 1$ pairs

Turn Over

2. For **ALL APPLICANTS.**

A list of real numbers $x_1, x_2, x_3, \ldots$ is defined by $x_1 = 1$, $x_2 = 3$ and then for $n \geqslant 3$ by

$$x_n = 2x_{n-1} - x_{n-2} + 1.$$

So, for example,

$$x_3 = 2x_2 - x_1 + 1 = 2 \times 3 - 1 + 1 = 6.$$

(i) Find the values of x_4 and x_5.

(ii) Find values of real constants A, B, C such that for $n = 1, 2, 3$,

$$x_n = A + Bn + Cn^2. \qquad (*)$$

(iii) Assuming that equation $(*)$ holds true for all $n \geqslant 1$, find the smallest n such that $x_n \geqslant 800$.

(iv) A second list of real numbers $y_1, y_2, y_3, \ldots$ is defined by $y_1 = 1$ and

$$y_n = y_{n-1} + 2n$$

Find, explaining your reasoning, a formula for y_n which holds for $n \geqslant 2$.

What is the approximate value of x_n / y_n for large values of n?

i) $x_4 = 2x_3 - x_2 + 1$
$= 2 \times 6 - 3 + 1$
$= 10$

$x_5 = 2x_4 - x_3 + 1$
$= 2 \times 10 - 6 + 1$
$= 15$

ii) $A + B + C = 1$ (1) $A + 2B + 4C = 3$ (2) $A + 3B + 9C = 6$ (3)

(2) − (1)
$B + 3C = 2$

(3) − (2)
$B + 5C = 3$

$2 - 3C = 3 - 5C$
$2C = 1$
$C = \frac{1}{2}$, $B = \frac{1}{2}$

sub into (1)
$A = 0$

$x_n = \frac{1}{2}n^2 + \frac{1}{2}n$

iii) $\frac{1}{2}n^2 + \frac{1}{2}n \geqslant 800$
$n^2 + n - 1600 \geqslant 0$
$n \geqslant \frac{-1 \pm \sqrt{1 + 6400}}{2}$ $\sqrt{6400} = 80$,
$n \geqslant \frac{-1 + 80}{2}$ (n is positive)
$n \geqslant 39.5$ smallest value of $n = 40$

iv) $y_1 = 1$ $y_2 = 1 + (2 \times 2)$ $y_3 = 1 + 4 + 2 \times 3$ $y_n = 1 + 4 + 6 + \ldots + 2n$

$y_n = 1 + 2(2 + 3 + \ldots + n)$ } arithmetic series

$y_n = 1 + 2 \times \frac{1}{2}(n-1)(2+n)$

$= 1 + 2n - n - 2 + n^2$

$= n^2 + n - 1$

$\frac{x_n}{y_n} = \frac{\frac{1}{2}(n^2+n)}{n^2+n-1}$ as n increases, $\frac{x_n}{y_n} \approx \frac{\frac{1}{2}(n^2+n)}{n^2+n} \approx \frac{1}{2}$

Turn Over

3.

For **APPLICANTS IN** $\left\{\begin{array}{l}\textbf{MATHEMATICS}\\ \textbf{MATHEMATICS \& STATISTICS}\\ \textbf{MATHEMATICS \& PHILOSOPHY}\\ \textbf{MATHEMATICS \& COMPUTER SCIENCE}\end{array}\right\}$ **ONLY.**

Computer Science **applicants should turn to page 14.**

For a positive whole number n, the function $f_n(x)$ is defined by

$$f_n(x) = \left(x^{2n-1} - 1\right)^2.$$

(i) On the axes provided opposite, sketch the graph of $y = f_2(x)$ labelling where the graph meets the axes.

(ii) On the same axes sketch the graph of $y = f_n(x)$ where n is a large positive integer.

(iii) Determine

$$\int_0^1 f_n(x)\, \mathrm{d}x.$$

(iv) The *positive* constants A and B are such that

$$\int_0^1 f_n(x)\, \mathrm{d}x \leqslant 1 - \frac{A}{n+B} \quad \text{for all } n \geqslant 1.$$

Show that

$$(3n-1)(n+B) \geqslant A(4n-1)n,$$

and explain why $A \leqslant 3/4$.

(v) When $A = 3/4$, what is the smallest possible value of B?

i) $f_2(x) = (x^3-1)^2 = y = x^6 - 2x^3 + 1$

$x=1,\ y=0$ $\quad x=-1,\ y=4$

$y=1,\ x=0$

$\frac{dy}{dx} = 6x^5 - 6x^2 = 0$

$6x^2(x^3-1)=0$

gradient $=0$ at $x=1,\ x=0$

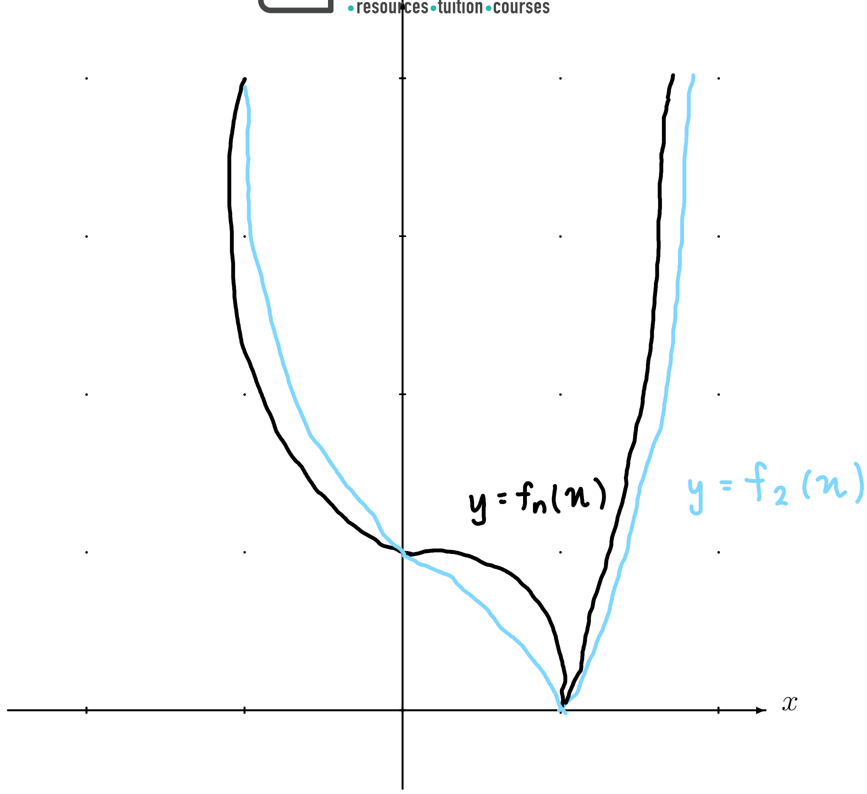
$f_n(x) = (x^{2n-1}-1)^2 = y$

$x=1,\ y=0$ $\quad x=-1,\ y=4$

$x=0,\ y=1$

For $-1<x<1$, when n is large $x^{2n-1} \approx 0$ and $f_n(x) \approx (-1)^2$

For $x>1$, as x increases, $(x^{2n-1}-1)^2$ (where n is large) increases at a faster rate than $(x^3-1)^2$.



iii) $\int_0^1 f_n(x)\,dx = \int_0^1 (x^{4n-2} + 1 - 2x^{2n-1})\,dx$

$= \left[\frac{x^{4n-1}}{4n-1} + x - \frac{x^{2n}}{n} \right]_0^1$

$= \left(\frac{1}{4n-1} + 1 - \frac{1}{n} \right) - 0$

$= \frac{n-(4n-1)}{n(4n-1)} + 1$

$= \frac{1-3n}{n(4n-1)} + 1$

iv) $1 - \frac{3n-1}{n(4n-1)} \leq 1 - \frac{A}{n+B}$

$\frac{3n-1}{n(4n-1)} \geq \frac{A}{n+B}$

$(3n-1)(n+B) \geq An(4n-1)$

$3n^2 - n + 3Bn - B \geq 4An^2 - An$

$(3-4A)n^2 + (A-1+3B)n - B \geq 0$

The coefficient of $n^2 \geq 0$, in order for the inequality to be valid for when n is large

$3 - 4A \geq 0$

$\therefore A \leq \frac{3}{4}$

v. $A = \frac{3}{4}$ $(3B - \frac{1}{4})n - B \geq 0$ $(3B - \frac{1}{4})n - B$ is at a minimum when $n = 1$

$2B - \frac{1}{4} \geq 0$

$B \geq \frac{1}{8}$ $\therefore$ smallest possible value of B is $\frac{1}{8}$

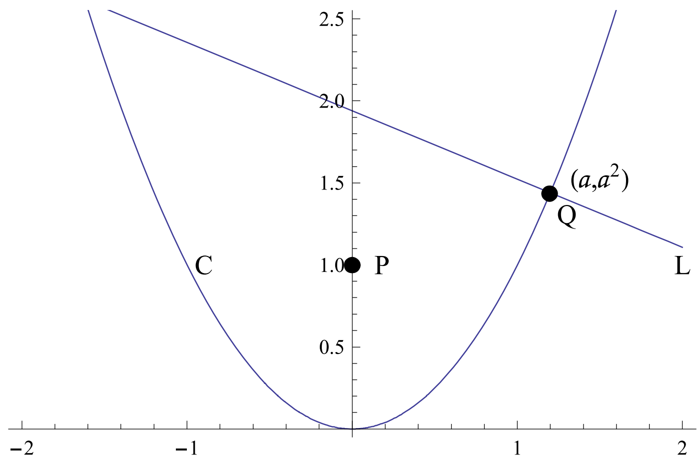
Turn Over

4.

For **APPLICANTS IN** $\left\{\begin{array}{l}\textbf{MATHEMATICS}\\ \textbf{MATHEMATICS \& STATISTICS}\\ \textbf{MATHEMATICS \& PHILOSOPHY}\end{array}\right\}$ **ONLY.**

Mathematics & Computer Science **and** *Computer Science* **applicants should turn to page 14.**

As shown in the diagram below: C is the parabola with equation $y = x^2$; P is the point $(0, 1)$; Q is the point (a, a^2) on C; L is the normal to C which passes through Q.



(i) Find the equation of L.

(ii) For what values of a does L pass through P?

(iii) Determine $|QP|^2$ as a function of a, where $|QP|$ denotes the distance from P to Q.

(iv) Find the values of a for which $|QP|$ is smallest.

(v) Find a point R, in the xy-plane but not on C, such that $|RQ|$ is smallest for a unique value of a. Briefly justify your answer.

4i. $y = x^2$ gradient of normal $= -\frac{1}{2x} = \frac{-1}{2a}$

$\frac{dy}{dx} = 2x$ $(x = a)$ $(y = a^2)$

$y - a^2 = \frac{-1}{2a}(x - a)$

$2ay - 2a^3 = -x + a$

$2ay + x = 2a^3 + a$

ii. $x = 0,\ y = 1$

$2a + 0 = 2a^3 + a$

$0 = a(2a^2 - 1)$

$a = 0,\ a = \sqrt{\frac{1}{2}},\ a = -\sqrt{\frac{1}{2}}$

iii. $|QP|^2 = (a - 0)^2 + (a^2 - 1)^2$

$|QP|^2 = a^2 + a^4 - 2a^2 + 1$

$|QP|^2 = a^4 - a^2 + 1$

iv. $\frac{d}{dx}(a^4 - a^2 + 1) = 0$ at minimum

$4a^3 - 2a = 0$

$2a(2a^2 - 1) = 0$

$a = 0,\ a = \sqrt{\frac{1}{2}},\ a = -\sqrt{\frac{1}{2}}$

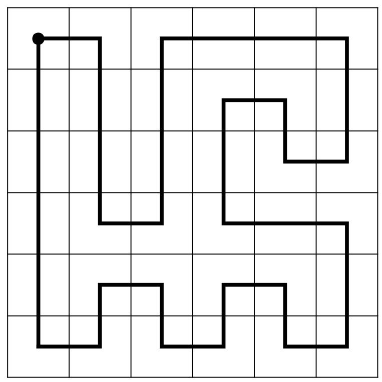
v. $R(0, -1)$, when $a = 0$. $(0, 0)$ is the minimum point on C and therefore the closest point on C to any point on the negative y-axis.

5. For **ALL APPLICANTS.**

Given an $n \times n$ grid of squares, where $n > 1$, a *tour* is a path drawn within the grid such that:

- along its way the path moves, horizontally or vertically, from the centre of one square to the centre of an adjacent square;
- the path starts and finishes in the same square;
- the path visits the centre of every other square just once.

For example, below is a tour drawn in a 6×6 grid of squares which starts and finishes in the top-left square.



For parts (i)-(iv) it is assumed that n is *even*.

(i) With the aid of a diagram, show how a tour, which starts and finishes in the top-left square, can be drawn in any $n \times n$ grid.

(ii) Is a tour still possible if the start/finish point is changed to the centre of a different square? Justify your answer.

Suppose now that a robot is programmed to move along a tour of an $n \times n$ grid. The robot understands two commands:

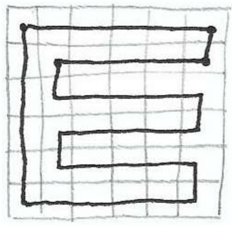
- command R which turns the robot clockwise through a right angle;
- command F which moves the robot forward to the centre of the next square.

The robot has a program, a list of commands, which it performs in the given order to complete a tour; say that, in total, command R appears r times in the program and command F appears f times.

(iii) Initially the robot is in the top-left square pointing to the right. Assuming the first command is an F, what is the value of f? Explain also why $r + 1$ is a multiple of 4.

(iv) Must the results of part (iii) still hold if the robot starts and finishes at the centre of a different square? Explain your reasoning.

(v) Show that a tour of an $n \times n$ grid is not possible when n is odd.

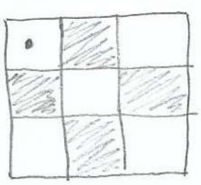
5i. 

As n is an even number, this pattern could work for any n – there would be $\frac{n}{2}$ branches, before returning upwards to the original point.

ii. Yes. The tour visits all squares so the same pattern can be traced from any point.

iii. $f = n^2$ (as every square is visited once and there are n^2 squares)
The robot starts pointing right and finishes pointing up (r turns). If it turns once more ($r+1$ turns), it will be pointing right again and will have taken a multiple of 4 to turn full circle to the original direction. $\therefore$ $r+1$ is a multiple of 4.

iv. f will be the same, as every square is still visited once. $r+1$ is not necessarily a multiple of 4 as the robot could start and finish pointing in the same direction.

v. 

The grid can be shaded in this way. If the robot starts on a light/dark square, it must finish on the same square (which is the same colour).
The robot would move in the pattern:
dark, light, dark, light, ... , dark
To move from one square to a square of the same colour requires an even number of moves. However, n is odd and n^2 (the number of moves) is also odd. $\therefore$ such a tour is impossible.

6.

For **APPLICANTS IN** $\left\{ \begin{array}{l} \textbf{COMPUTER SCIENCE} \\ \textbf{MATHEMATICS \& COMPUTER SCIENCE} \end{array} \right\}$ **ONLY.**

(i) Alice, Bob, and Charlie make the following statements:

Alice: Bob is lying.
Bob: Charlie is lying.
Charlie: $1 + 1 = 2$.

Who is telling the truth? Who is lying?. Explain your answer.

(ii) Now Alice, Bob, and Charlie make the following statements:

Alice: Bob is telling the truth.
Bob: Alice is telling the truth.
Charlie: Alice is lying.

What are the possible numbers of people telling the truth? Explain your answer.

(iii) They now make the following statements:

Alice: Bob and Charlie are both lying.
Bob: Alice is telling the truth or Charlie is lying (or both).
Charlie: Alice and Bob are both telling the truth.

Who is telling the truth and who is lying on this occasion? Explain your answer.

6i. 1+1 does equal 2, so Charlie is telling the truth. This means that Bob is lying, which means Alice is telling the truth.

ii. If Alice is telling the truth:
Bob is also telling the truth, while Charlie is lying.
If Alice is lying:
Bob is also lying, while Charlie is telling the truth.
Therefore either 2 people are telling the truth or 1 person is telling the truth.

iii. If Charlie is telling the truth:
Alice's statement that Charlie is lying must be true, which is impossible. Therefore Charlie is lying and at least one of Alice/Bob is also lying.
Since Charlie is lying, Bob is telling the truth and Alice's statement that both Bob and Charlie are lying is false.
Therefore, Charlie and Alice are lying, Bob is telling the truth.

7. For **APPLICANTS IN COMPUTER SCIENCE ONLY.**

Consider sequences of the letters M, X and W. *Valid* sequences are made up according to the rule that an M and a W can never be adjacent in the sequence. So M, XMXW, and XMMXW are examples of valid sequences, whereas the sequences MW and XWMX are not valid.

(i) Clearly, there are 3 valid sequences of length 1. List all valid sequences of length 2.

(ii) Let $g(n)$ denote the number of valid sequences of length n. Further, let $m(n)$, $x(n)$, $w(n)$ denote the number of valid sequences of length n that start with an M, an X, a W respectively.

Explain why

$$\begin{aligned} m(n) &= w(n), \\ m(n) &= m(n-1) + x(n-1) \quad \text{for } n > 1, \\ x(n) &= 2m(n-1) + x(n-1) \quad \text{for } n > 1, \end{aligned}$$

and write down a formula for $g(n)$ in terms of $m(n)$ and $x(n)$.

Hence compute $g(3)$, and verify that $g(4) = 41$.

(iii) Given a sequence using these letters then we say that it is *reflexive* if the following operation on the sequence does not change it: reverse the letters in the sequence, and then replace each occurrence of M by W and vice versa. So MXW, WXXM and XWXMX are reflexive strings, but MXM and XMXX are not. Let $r(n)$ be the number of valid, reflexive sequences of length n.

If a sequence is reflexive and has odd length, what must the middle letter be? Explain your answer.

Hence, show that

$$r(n) = \begin{cases} x\left(\frac{n+1}{2}\right) & \text{if } n \text{ is odd}, \\ x\left(\frac{n}{2}\right) & \text{if } n \text{ is even}. \end{cases}$$

7i. MM, MX, XX, XM, XW, WW, WX

ii. By replacing all "M"s with "W"s and all "W"s with "M"s in a list of valid sequences of length n beginning with M, you will get the list of valid sequences of length n beginning with W (and vice versa) $\therefore m(n) = w(n)$

A valid sequence of length n beginning with M is formed by adding M to the front of a valid sequence of length n-1. As M cannot be adjacent in W, M can only be added to sequences starting with an M or an X. $\therefore m(n) = m(n-1) + x(n-1)$

A valid sequence of length n beginning with X is formed by adding X to the front of a valid sequence of length n-1.

$\therefore x(n) = m(n-1) + w(n-1) + x(n-1)$

as $m(n-1) = w(n-1)$, $x(n) = 2m(n-1) + x(n-1)$

$g(n) = x(n) + m(n) + w(n) \quad (m(n) = w(n))$

$g(n) = 2m(n) + x(n)$

$m(2) = 2 \quad m(3) = m(2) + x(2) = 2 + 3 = 5$

$x(2) = 3 \quad x(3) = 2m(2) + x(2) = 2\times2 + 3 = 7$

$g(3) = 2m(3) + x(3) = 2\times5 + 7 = 17$

$g(4) = 2m(4) + x(4)$

$m(4) = m(3) + x(3) = 5 + 7 = 12$

$x(4) = 2m(3) + x(3) = 2\times5 + 7 = 17$

$g(4) = 2\times12 + 17 = 41$

iii. The middle letter is X if it is to remain unchanged. M or W would be replaced and the sequence would not be reflexive.

For an odd sequence: Take a sequence beginning with X, reflect it and replace each occurrence of M by W and vice versa. Add this to the beginning of the original sequence, removing one of the middle "X"s. $\therefore r(n) = x\left(\frac{n+1}{2}\right)$, if n is odd.

For an even sequence: Take a sequence beginning with X, reflect it and replace each occurrence of M by W and vice versa. Add this to the beginning of the original sequence.

$\therefore r(n) = x\left(\frac{n}{2}\right)$, if n is even.